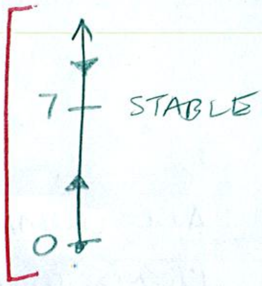
A population of rare fish lives only in one secluded lake. The population follows a logistic growth model, with the total number of fish satisfying the differential equation $\frac{dP}{dt} = \frac{1}{2}P(7-P)$,
where t is measured in years, and P is measured in thousands of fish.
Answer the following questions without solving algebraically for $P(t)$.

SCORE: _____ / 8 PTS

[a] If this situation continues indefinitely ("forever"), what will be the ultimate population of fish in the lake ?
Specify the units of your answer.

2



$\frac{dP}{dt} = 0 \longrightarrow P = 0, 7$

1 7000 FISH

[b] A fishing club has discovered the lake, and are removing the fish at a rate of 3000 fish per year.

[i] Write a differential equation for the total number of fish in this new situation.

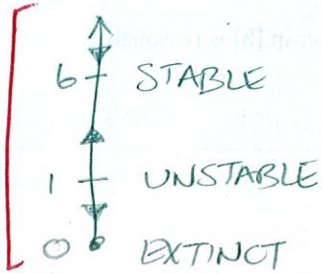
1 $\frac{dP}{dt} = \frac{1}{2}P(7-P) - 3$

$= \frac{1}{2}(-P^2 + 7P - 6)$

$= -\frac{1}{2}(P^2 - 7P + 6) = -\frac{1}{2}(P-1)(P-6)$

[ii] What is the minimum number of fish that must initially be in the lake in order to prevent the population from going extinct in this new situation ?

2



1 1000 FISH

[iii] If this new situation continues indefinitely ("forever") without the population going extinct, what will be the ultimate population of fish in the lake ?

1 6000 FISH

A 1600 liter tank initially holds 400 liters of brine containing 5 grams of salt per liter. Brine containing 2 grams of salt per liter starts flowing into the tank at 12 liters per minute. At the same time, the well-mixed solution leaves the tank at 4 liters per minute.

SCORE: _____ / 22 PTS

[a] Find the amount of salt in the tank t minutes after the less concentrated brine starts to enter the tank (but before the tank starts to overflow). **HINT: Simplify all fractions as soon as possible.**

4

$$\frac{dA}{dt} = 12\cdot 2 - 4\cdot\frac{A}{400+(12-4)t} = 24 - \frac{A}{100+2t}, \quad A(0) = 400\cdot 5 = 2000$$

$$\frac{dA}{dt} + \frac{A}{100+2t} = 24$$

$$\mu = e^{\int \frac{1}{100+2t}\,dt} = e^{\frac{1}{2}\ln|100+2t|} = (100+2t)^{\frac{1}{2}}$$

$$(100+2t)^{\frac{1}{2}}\frac{dA}{dt} + (100+2t)^{-\frac{1}{2}}A = 24(100+2t)^{\frac{1}{2}}$$

CHECK: $\frac{d}{dt}(100+2t)^{\frac{1}{2}} = \frac{1}{2}(100+2t)^{-\frac{1}{2}}\cdot 2 = (100+2t)^{-\frac{1}{2}}$ ✓

$$(100+2t)^{\frac{1}{2}}A = 24\cdot\frac{2}{3}\cdot\frac{1}{2}(100+2t)^{\frac{3}{2}} + C$$

$$= 8(100+2t)^{\frac{3}{2}} + C$$

$$A = 8(100+2t) + C(100+2t)^{-\frac{1}{2}}$$

$$2000 = 8(100) + C(100)^{-\frac{1}{2}} = 800 + \frac{C}{10}$$

$$C = 12000$$

$$A = 8(100+2t) + 12000(100+2t)^{-\frac{1}{2}}$$

ALL UNDERLINED ITEMS WORTH 1 POINT UNLESS OTHERWISE NOTED

[b] Find the concentration of salt in the tank at the instant the tank starts to overflow.

$$400 + 8t = 1600 \rightarrow t = 150$$

$$\frac{A(150)}{1600} = \frac{8(400) + 12000(400)^{-\frac{1}{2}}}{1600} = \frac{32 + \frac{120}{20}}{16} = \frac{38}{16} = \frac{19}{8}\ \frac{g}{L}$$

$$= 2\tfrac{3}{8}\ \frac{g}{L}$$

[c] **Without referring to the differential equation you wrote in [a],** explain why your answer in [b] is reasonable. Your answer may involve any numbers from the original description of the situation.

THE CONCENTRATION IS BETWEEN THE INITIAL CONCENTRATION ($5\frac{g}{L}$) AND THE INCOMING CONCENTRATION ($2\frac{g}{L}$). IT IS CLOSER TO THE INCOMING CONCENTRATION BECAUSE A LOT OF "NEW" BRINE HAS ENTERED THE TANK AFTER 150 MINUTES.

[d] Write, **but do NOT solve,** an initial value problem for the amount of salt in the tank t minutes after the tank starts to overflow.

3

$$\frac{dA}{dt} = 12\cdot 2 - 12\cdot\frac{A}{1600} = 24 - \frac{3A}{400}$$

$$A(0) = 3800$$